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CONVECTION IN STARS[1]

I. Basic Boussinesq Convection[2]

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Convection occurs somewhere in most stars, yet our lack of understanding of convection has not seemed a major impediment to progress in stellar structure in recent years. In part this is true because convection often achieves the idealized adiabatic limit that is expected in convective cores of stars. It has also been true that uncertainties in the other physical processes in stars have been reduced considerably, and this has permitted a better empirical determination of the arbitrary parameters used in stellar convection theory. Of course, there is always the possibility that things are not as satisfactory as one thinks. But if we take the optimistic view that present convective models are qualitatively reasonable, what can one expect of an improved theory? One desirable feature would be the prediction of convective transfer with, in addition, some reasonable estimate of the accuracy of the prediction. For this, a minimal but inadequate test is found in laboratory convection for which some quantitative data are available. Thus, a principal goal of stellar convection theory should be the development of a reasonable deductive theory whose reasonability can be minimally established by laboratory tests.

Having obtained a theory at this level we would next be interested in finer details that characterize stellar convection. That is, we would like to be able to be quantitative about the time dependence and scales of the convection motion and to compare these with solar observations; we would like to know how far convection may penetrate beyond the regions of instability and by large-scale mixing remove chemical inhomogeneities; we would be interested in the precise temperature variations at the tops of convective envelopes to have better input for model atmospheres. And these are only a sample of some of the questions that one would hope to answer at this level of difficulty.

There is, in addition, a series of dynamical questions which raise problems about the interaction of convection with other processes of stellar fluid dynamics. These bring in new instabilities and are probably the most in-

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teresting problems in stellar convection theory at present. Thus we would like to know when convection can stabilize or destabilize pulsation; we would like to understand the role of convection in the rotational history of the Sun; we would like, even in a primitive example, to compute the dynamo effect of rotation and convection from first principles.

Before we can proceed to a discussion of what is known about these various processes, we need to have an outline of the basic theoretical techniques of modern convection theory, some of which involve prodigious calculations. For the most part, these techniques have been developed and tested on the basic problem of convection in a thin-plane layer of fluid. Accordingly, Part I of this review is devoted to the problem of convective transfer in the laboratory situation. It should be stressed, however, that the equations and approximations used are the same as those now used in stellar structure calculations with a common goal—to predict the march of temperature through the convective fluid. It will be seen, for example, that the mixing-length theory as used in stars is not as complete as that now discussed for laboratory convection. Other more difficult, but hopefully more adequate, approaches will be outlined. However, the astrophysicist interested in a simple recipe for calculating stellar models will be disappointed to find that instead, the stellar structure calculation in these approaches constitutes a subroutine in the convection program, and not vice versa. There seems no way around this for the present.

Having outlined these various approaches here, we shall return in the next volume of these *Reviews* to their application to the problem of stellar convection. In Part II the special problems of stellar convection such as large density variation, overshooting, rotation, and radiative transfer will be considered in the context of pure convection theory. It is in these domains that the more involved methods, especially those of Section 8, can be used to advantage, though one would not wish to pursue them without first examining their suitability for the basic problem of convective transfer. Part III is then to be devoted to actual stellar convection and the understanding of it that can be drawn from the discussion of Part II.

It is hardly necessary to add that even the limited subject matter of Part I cannot be treated exhaustively in the space available here. Hence, the discussion is focused on approaches that seem to have a direct bearing on the problems of stellar convection. Since the literature is vast, no attempt is made to cover all the contributions from meteorology (Sutton 1953, Priestley 1959), engineering (Prandtl 1952) and other fields where convection plays an important role. Also, reference is often made to papers which adequately summarize or synthesize previous work, and fundamental papers covered in such discussions are not necessarily cited. Particularly helpful is the thoroughness of existing treatments of linear theory (Chandrasekhar 1961), and the existence of a recent general review of the subject (Brindley 1967), in addition to some less complete ones by the present author (Spiegel

1966, 1967). An interesting discussion of the overall spectral dynamics of convection is also available (Platzman 1965).

1. The Equations of Convection

1.1 *The anelastic and Boussinesq approximations.*—Though it would be out of place here to go into mathematical details of convection theory, it is necessary to discuss the basic equations, since much of the language of the subject stems from them. However, it does not pay to write down the full equations, since no attempt seems to have been made to solve them, except in linear theory. It is more usual to begin at the outset by introducing approximations.

The approximation that seems most appropriate for astrophysical convection is the anelastic approximation familiar to meteorologists (Ogura & Phillips 1962, Gough 1969). The basic idea of this approximation is to filter out high-frequency phenomena such as sound waves since these are thought to be unimportant for transport processes. This approximation is not really valid in the outer layer of convective envelopes or red giants, for example, since the Mach numbers of the convective motions can become appreciable there; but since even the anelastic problem has not been solved for that case, little can be said of this difficulty.

In studying laboratory convection, a further approximation is permitted, namely that the vertical extent of the fluid is much less than its density or pressure scale heights (Spiegel & Veronis 1960). This does not mean that the fluid is incompressible, but it does imply that density variations are very small and permits other such simplifications (Mihaljan 1962, Malkus 1964). This approximation combined with the anelastic approximation leads to the so-called *Boussinesq approximation* which is used in studies of laboratory convection, meteorology [sometimes with other justifications (Dutton & Fichtl 1969)], and even (implicitly) in most calculations of stellar convection. It often is further assumed that material properties such as viscosity and conductivity are insensitive to temperature; that is not an essential part of the Boussinesq approximation, but this "strong" form of the approximation is adequate for many experiments. Further, the principal configuration studied is that of a plane-parallel layer of fluid oriented horizontally in a uniform gravitational field, and that example will serve here.

1.2 *Mean quantities.*—In a convecting fluid, and especially a turbulent one, it is convenient to separate the mean and fluctuating parts of variables such as pressure and temperature. The means should ideally be ensemble averages, but it is computationally more convenient to use means over horizontal surfaces. Thus, one writes for the temperature in the plane-parallel case,

$$T(\mathbf{x}, t) = \overline{T}(z, t) + \theta(\mathbf{x}, t) \qquad 1.1$$

where z is the vertical coordinate and $\bar{\theta}=0$. In the idealized problem of Part I, $\bar{\mathbf{u}}=0$ where $\mathbf{u}$ is the velocity.

When horizontal means are taken of the Boussinesq equations there result (Spiegel 1967)

$$\frac{\partial}{\partial z}(\bar{p} + \rho\overline{w^2}) = -g\rho \qquad 1.2$$

and

$$\frac{\partial}{\partial t}\overline{T} + \frac{\partial}{\partial z}\overline{w\theta} = \kappa\frac{\partial^2\overline{T}}{\partial z^2} \qquad 1.3$$

Here ρ is the density (assumed constant), p is the pressure, g is the acceleration of gravity, w is the vertical component of $\mathbf{u}$, and κ is the thermal diffusivity (molecular or radiative). These two equations are the Boussinesq versions of two of the basic equations of stellar structure theory. The $\rho\overline{w^2}$ is the turbulent pressure and $\overline{w\theta}$ is the convective flux. If these two terms could be simply evaluated in terms of mean quantities, the convective difficulties of stellar structure theory would be essentially overcome, since a reasonably simple set of equations would result. No such possibility is readily found from the equations for $\mathbf{u}$ and θ (which will be displayed below). Indeed, it is clear from looking at the full equations that static structure equations like 1.2 and 1.3 make up one of the least difficult parts of the convection problem.

1.3 *The Boussinesq equations.*—In writing the equations of motion it is advantageous to use natural units appropriate to the problem. Thus we take the vertical extent of the fluid d as the unit of length, d^2/κ as the unit of time, where κ is the thermal diffusivity, ρ as unit of density, and $\Delta T - gd/C_p$ as the unit of temperature, where ΔT is an imposed temperature difference across the fluid, g is the acceleration of gravity, and C_p is the specific heat at constant pressure. The term gd/C_p is the adiabatic temperature change across the layer and meteorologists would call $\Delta T - gd/C_p$ the change in potential temperature (Brunt 1939), but astrophysicists prefer to work with entropy.

In natural units, the equations for the fluctuating quantities are (Malkus & Veronis 1958)

$$\frac{1}{\sigma}\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla\varpi + R\hat{\theta}\mathbf{k} + \nabla^2\mathbf{u} \qquad 1.4$$

$$\frac{\partial\theta}{\partial t} + w\left(\frac{\partial\overline{T}}{\partial z} - \beta_A\right) + \mathbf{u}\cdot\nabla\theta - \overline{\mathbf{u}\cdot\nabla\theta} = \nabla^2\theta \qquad 1.5$$

$$\nabla \cdot \mathbf{u} = 0 \qquad 1.6$$

where $\hat{\mathbf{k}}$ is a unit vector in the vertical, $\varpi = (p - \bar{p})/\rho - \overline{w^2}$,

$$R = \frac{g\alpha d^3}{\kappa\nu}\left(\Delta T - \frac{gd}{C_p}\right), \qquad \sigma = \nu/\kappa \qquad 1.7$$

are known as the Rayleigh and Prandtl numbers, and α is the coefficient of thermal expansion.

These equations are completed with the addition of 1.2 and 1.3, but the former is not really essential for the Boussinesq case. Moreover, 1.3 can be simplified by adopting the widely held belief that under stationary external conditions mean quantities are also stationary, at least in turbulent convection. In that case, 1.3 can be written, in natural units, as

$$\overline{w\theta} - \frac{\partial \overline{T}}{\partial z} = N \qquad 1.8$$

where the Nusselt number N is the (constant) sum of the convective and conductive heat fluxes. If the Boussinesq approximation had not been made, additional terms representing acoustic flux and transport due to viscous stresses would be required, though such effects are usually ignored in stellar convection as well.

To these equations must be added boundary conditions (Chandrasekhar 1961). In experimental studies the attempt is often made to fix the boundary temperatures, hence $\theta = 0$ on the boundaries. On rigid boundaries, $\mathbf{u} = 0$, and on free boundaries $w = 0$ and the tangential stresses vanish. The conditions appropriate to free boundaries are often used in theoretical work even when they do not apply, since they are easier to work with.

2. Stability Theory

Convection as we are considering it here normally arises as an instability which grows on a previously static configuration. In the simplest case, where a perturbation of infinitesimal amplitude does not suffer thermal diffusion or viscous effects, the question of stability can be decided simply. A parcel of fluid displaced vertically and adiabatically suffers a change in energy by an amount $mC_p dT + mg dz$ where m is the mass of the parcel. If this change is negative, that is if

$$\frac{dT}{dz} < -g/C_p \qquad 2.1$$

we can expect instability. This criterion, called the *Schwarzschild criterion*, can be obtained by more rigorous methods (Lebovitz 1966, Kaniel & Kovitz 1967).

When dissipative effects on the perturbation are included, the problem of stability is more difficult but precise treatments are available (Chandrasekhar 1961). An infinitesimal perturbation is applied to a given static configuration. For the initial times at least, the equations of motion are then linear, with constant coefficients in the strong Boussinesq approximation, and they are then separable. In the plane-parallel case solutions of the form

$$w(\mathbf{x}, t) = f(x, y)e^{\eta t}W(z) \qquad 2.2$$

for w, are found with similar forms for the other variables. Here

$$\nabla_1^2 f \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = - a^2 f \qquad 2.3$$

and η and a are separation constants representing the growth rate and horizontal wavenumber of the perturbation. Solutions of the resulting eigenvalue problem can be found giving η as a function of R, σ, and a. If for given R and σ there exist solutions with $\mathrm{Re}(\eta) > 0$ for some a, the solution is unstable. If for $\mathrm{Re}(\eta) \geq 0$ we have $\mathrm{Im}(\eta) \neq 0$ the instability is called overstability or vibrational instability. If $\mathrm{Im}(\eta) = 0$ whenever $\mathrm{Re}(\eta) \geq 0$, the principle of the exchange of stabilities is said to hold. For the linearized form of 1.4–1.6 this principle has been established. If we set $\eta = 0$ in the separated linear equations they give a relation defining the condition for marginal stability. As can be seen by inspection, σ drops out of these equations and we are left with a relation between R and a, having the properties $R \to \infty$ for $a \to 0$ and $a \to \infty$. Thus R has a minimum value R_c at a particular $a = a_c$, which implies that for $R > R_c$ there is a band of a for which there exist $\eta > 0$. We conclude that for $R > R_c$ convection occurs. For rigid boundaries the values $R_c = 1708$ and $a_c = 3.1$ are found. A physical discussion of these results suggests that the Rayleigh number may be interpreted as the ratio of buoyancy force to viscous force on the perturbation (Spiegel 1960).

Now stability is much harder to establish than instability, since it is not always possible to test all possible perturbations. For the present elementary example of convection it has also been shown that for $R > R_c$ there are no positive η, but this is not always the case in more complex problems, and overstability may arise if stabilizing forces work against the convection. The overstability then may occur at $R < R_c$ in the sense that R_c is defined here.

Further, some of the more complicated configurations, though stable in the sense used here, may be metastable. That is, if perturbations of sufficient amplitude are introduced, the system does not return to its initial configuration. In fluid dynamics a metastable system is said to exhibit finite amplitude instability. Systems that exhibit overstability often are metastable as well and several examples will arise in Part II.

These various possibilities are discussed here to make clear that the example we are considering is special in that instability occurs only as exponential growth when $R > R_c$ (Sani 1964). But it should also be stressed that when the fluid is unstable there is in general an infinitude of stable modes lying in a continuous band of horizontal wavenumber a. A further degeneracy is that associated with the various possible f's that satisfy 2.2 (Bisshopp 1960).

3. Experimental Results

Most of Part I of this review consists of discussion of various attempts to understand Boussinesq convection and in particular to find how N depends on R and σ. Some astronomers do not consider that this topic is necessarily relevant to stars since the Boussinesq approximation does not hold in stars. However, as noted in Section 1, the Boussinesq equations are a limiting case of the equations governing stellar convection. Thus, any scheme for solving the equations of stellar convection should also work in the Boussinesq approximation or be subject to grave doubts. Whether a method does work in this limit can be tested only by experiment (good numerical experiments may have to suffice) over as wide a range of parameters as possible. Of course, such checks do not by any means guarantee the validity of a stellar convection theory, but they consitute a basic and fairly exacting requirement. Moreover, as we shall see in Section 8, the mixing-length theory now used for stars is a Boussinesq theory.

The onset of convective motions at a critical value of R is well established; the measured critical value is usually within 3% of that given by stability theory or better (Thompson & Sogin 1966) and, as theory predicts, R_c is independent of σ. For R just above R_c, steady cellular motion is observed for $\sigma \gtrsim .7$ and if enough care is taken, the motion is steady and occurs in two-dimensional patterns called *rolls*. The widely quoted remark that hexagonal patterns occur in steady convection is not borne out by modern experiments, except under special circumstances, as when the fluid has properties which are temperature dependent and the strong Boussinesq approximation is not valid (Tippleskirch 1956). The wavenumber of the rolls decreases as R increases (Koschmieder 1966, 1969, Krishnamurti 1970) with a rate which depends on σ but for which no simple experimental relation has as yet been given; this behavior has been attributed to the side boundaries (Davis 1968), but recent numerical experiments without sidewalls show this behavior (Lipps & Somerville 1971). The pattern of the motion changes when R is raised above 22,600 (for $\sigma \geq 1$) and becomes three-dimensional. The preferred form then seems rectangular, and probably consists of crossed rolls (Busse 1970, Busse & Whitehead 1971). Such steady patterns persist to Rayleigh numbers $\sim 5 \times 10^4$ when the motion becomes time dependent, but periodic. Finally at higher values of $R(> 10^6)$ the motion becomes aperiodic, and with even higher R it probably becomes turbulent if σ is not too large. The various transitions in the nature of the flow are currently the object of intense interest (Krishnamurti 1970, Willis & Deardorff 1967a, b, c; Chen & Whitehead 1968).

These results have been obtained for $\sigma \gtrsim 1$. For $\sigma < 1$ the data are too sparse to permit any real conclusions but the various transitions appear to occur at rather reduced values of R (Krishnamurti 1970). (In laboratory fluids typical values of σ are: for silicone oils$\gg$1, for water = 7, for air = .7, for mercury = .025, for liquid sodium = .005.) What is remarkable is the marked change in behavior that occurs in the neighborhood of $\sigma = 1$. In particular for air, transition to time-dependent convection already occurs at $R = 5 \times 10^3$, before the transition to three-dimensional motion (Willis & Deardorff 1970). For mercury, steady motion is not found at all (Rossby 1969).

The kind of experiment that seems easiest to use in checking astrophysically oriented theories is the measurement of the heat transfer, or equivalently the Nusselt number, for different values of R and σ. Of course, the R and σ typical for stellar convection are not really accessible experimentally; in the Sun, for example, R is about 10^{12} to 10^{20} (depending on whether one takes d as a scale height or the depth of the convective zone and on how one chooses the other parameters) and $\sigma \sim 10^{-9}$ because the thermal diffusion is radiative (Ledoux, Schwarzschild & Spiegel 1961). Nevertheless, one should try to push to the highest possible R and the lowest possible σ experimentally to provide data for testing theories as stringently as possible. At present, the data are inadequate to really discriminate among various theories, but let us consider what information is available.

In general, the measured heat fluxes through a convective layer are steady in time. Thus the Nusselt number defined in 1.8 is a constant and one writes

$$J = \kappa \frac{\Delta T}{d} N(R, \sigma) \qquad 3.1$$

where J is the heat flux divided by ρC_p. Since the only parameters in the equations are R and σ, the assumption that N depends only on them is quite reasonable, though any failure of the theoretical boundary conditions to match the actual ones and any deviations from the strong Boussinesq approximations may spoil this simple functional form.

For $R \leq R_c$, $N = 1$, since convection does not occur. (Exceptions may arise when the fluid is metastable for $R \leq R_c$.) The slope of the curve of N vs R (for fixed σ) breaks from $N = 1$ at $R = R_c$ and N begins to increase with R (e.g. Rossby 1969). A remarkable feature of the experiments is that N is only a piecewise smooth function of R, and a series of breaks in the N-R curves are observed (Malkus 1954b, Krishnamurti 1970, Willis & Deardorff 1967b). The transitions are discontinuities in dN/dR like the one which signals the onset of convection, but the relative jumps are increasingly small as R increases. No transitions have been reported for $R > 5 \times 10^6$. One suggested explanation of the transitions is that they mark the onset of new modes of motion (Malkus 1954a) which become unstable in the convectively altered conditions of fluid. Another possibility is that some of the transitions represent the occurrence of new interactions among the already existing modes.

Most attempts to describe the measurements of N are based on the interpolation formula

$$N = AR^r\sigma^s \qquad 3.2$$

and experimentalists quote measured values for A, r, and s. Although the quoted uncertainties in several experiments are reasonable, there are some uncomfortably large differences among values given by different experimentalists in A, r, and s, and, less often, in the actual values of N itself. Part of the trouble in the experiments is that heat leaks out of the sides of the apparatus and the correction for this is uncertain. Another experimental difficulty is that to raise R and maintain the strong Boussinesq approximation one must raise d. To keep the same aspect ratio one must then increase the horizontal dimension of the apparatus. The result is a large volume of fluid which is thermally sluggish; such systems take long times to come to equilibrium and are hard to maintain at given boundary temperatures.

Yet another problem relates to data analysis: different workers fit 3.2 to the data in different domains of R, and if r has a weak R dependence, discrepancies are inevitable. Of course, if r depends on R this may imply that 3.2 is inadequate, but there are theoretical approaches suggesting that 3.2 works well in certain well-defined domains of R. If data which spread over more than one of these domains are fitted to 3.2 the results may be misleading. A careful analysis of these points is lacking.

Given these uncertainties we may note some trends in the data. The general impression is that above $R \sim 5\times10^5$, for $\sigma > 1$, there appears a marked but continuous increase in the upward variation of N with R. No data for $R > 3\times10^9$ seem available and for data in the range 5×10^6 to 3×10^9 the reported values of r range from 0.200 to 0.325 with quoted probable errors $\pm.005$ (Goldstein & Chu 1966, O'Toole & Silveston 1961, Rossby 1969, Sommerscales & Gazda 1969). The values of A associated with these extremes are from about 0.2 to 0.08. The most reliable estimates for r seem to be 0.30–0.33.

These differences may partly be due to Prandtl number dependence, since the value $r = .283$ was found in silicone oil ($\sigma \sim 20$) (Sommerscales & Gazda 1969) and .325 in acetone ($\sigma = 3.7$) (Malkus 1954a). Other data seem consistent with this possibility: for example in one experiment (Rossby 1969) $r = .30 \pm .005$ and $A = .13$ were found for water ($\sigma = 7$) using data down to $R = 4\times10^3$. However, limited attempts to find s suggest that it is small for $\sigma \geq 1$ (O'Toole & Silveston 1961), hence one picture that might be considered for $\sigma > 1$ is that $s \sim 0$ but A and r depend on σ in some transcendental way. An alternative interpretation is that the domain of R in which 3.2 is a good representation depends on σ and the apparent variation of A and r with σ is a result of fitting 3.2 to the data in inappropriate domains. This possibility is also implied by data for $R < 5\times10^5$ which give lower values of

r (.25–.28) (Rossby 1969), and it may be true that the inclusion of data from lower values of R may depress r. A conclusion widely drawn from these data is that when R becomes large enough, $N \propto R^{1/3}$; this is in keeping with an older suggestion that for $R \lesssim 5 \times 10^5$, $N \propto R^{1/4}$ and for $R \gtrsim 10^5$, $N \propto R^{1/3}$ (Jakob 1946). The modern data do not permit such a simple picture in detail, but support it qualitatively.

For $\sigma \leq 1$, it appears that N drops with σ more discernibly than at higher σ. Data on this point are available only for air ($\sigma = .7$) and mercury ($\sigma = .025$) and a definitive statement on the value of s or on possible σ dependence of N or r cannot yet be made.

The experimental information about N thus seems uncertain in all respects. Even the most recent measurements of N differ between experimentalists by 10% or more at the same R and σ. This level of uncertainty is far greater than is found in calorific measurements in modern physics and there is a definite need to increase the quantity and accuracy of data on convective heat transport. This would seem a useful subject to include in programs of laboratory astrophysics.

Another kind of measurement which has been made in some detail is of $\overline{T}(z)$ (Goldstein 1964; Rossby 1969, Sommerscales & Gazda 1969, Thomas & Townsend 1957, Townsend 1959, Willis & Deardorff 1967c). At large R it is found qualitatively that $\overline{T}$ is nearly constant away from the boundaries; near the boundaries $\overline{T}$ varies rapidly with z. (A variation in $\overline{T} \sim gd/C_p$ would be hard to detect in the laboratory.) From 1.8 and the boundary conditions, it is clear that $-\partial\overline{T}/\partial z = N$ at the boundaries, simply because the convective flux vanishes there. But, as to further details about $\overline{T}$, different experimentalists do not agree. Some results indicate that $\overline{T}$ is not symmetric about the midplane of the layer but others do not. The equations are invariant to reflection through the midplane but this does not necessarily imply that $\overline{T}$ is symmetric, since solutions may occur in asymmetric pairs. The different results may have to do with variations in experimental setup, non-Boussinesq effects, different measuring techniques (which mix some of θ into $\overline{T}$), different averaging times, and differing states of motion.

An interesting feature found in the experiments is that $\overline{T}$ is not always monotonic. In particular, there are often bumps in $\overline{T}$ just inside one or both of the thermal boundary layers (Sommerscales & Gazda 1969). These bumps are not universally accepted, but their existence now seems quite likely.

Other aspects of the motion, such as $\overline{\theta^2}$, are measured and other various external effects, such as rotation, have been studied. Many of these will be discussed in Part II.

4. Calculations for Mildly Supercritical R

If R is just above the critical value R_c for the onset of convection, there are straightforward analytical and numerical techniques for finding solutions of the basic equations (Joseph 1966, Kuo 1959, Kuo & Platzman 1961, Malkus & Veronis 1958, Schlüter, Lortz & Busse 1965, Segel 1966, Stuart

1958). Though this topic (known as *finite amplitude theory*) is the most active and successful branch of convection theory, it is not directly helpful to astrophysicists because in stars $R \gg R_c$ in general. Nevertheless, even in discussing what happens at high R, solutions at low R may give a qualitative lead, and a brief outline of the topic is in order here. The remaining sections on theories will then be devoted to large R.

The usual analytic technique often used when $R-R_c$ is small is a version of degenerate perturbation theory. The basic assumption is that for R slightly above R_c the motions will develop only small amplitude and a perturbation expansion in amplitude is made. Thus one writes $\mathbf{u}=\epsilon\mathbf{u}_0+\epsilon^2\mathbf{u}_1+\dots$, $\theta=\epsilon\theta_0+\epsilon^2\theta_1+\dots$, where $\mathbf{u}_0$ and θ_0 are normalized functions and the amplitude ϵ is assumed small. Though in convection problems it is usually assumed that R is prescribed, it is not known in advance which value of R will produce a particular amplitude. It is convenient therefore to consider ϵ as given and to find the R required to produce it; thus R can be thought of as a function of ϵ and we can write $R=R_0+\epsilon R_1+\dots$, with the understanding that this relation can later be inverted.

The leading terms in the expansion give the linear equations of stability theory. These are degenerate since an infinity of choices of the wavenumber a and the horizontal planform f can be made (cf Equation 2.3). The problem is usually restricted by choosing the solution at this stage to be one of the marginally stable solutions of linear theory, though recent work has been directed at time-dependent cases (Matkowsky 1970). Having made this choice one goes on to higher orders using known techniques of perturbation theory to suppress resonances. The expansions have been carried to high order, and are convergent (Lortz 1961); solutions good to $R \sim 10R_c$ or more are now routinely computable.

In the expansions carried out thus far, the linear solution which is perturbed is taken to have a single, fixed horizontal wavenumber a and fixed planform. Since this linear solution is marginally stable, R_0 is the value of R that produces marginal stability. If R_0 were much greater than R_c, there would exist values of a corresponding to highly unstable modes at $R=R_0$, and these would grow to large amplitude and overwhelm the solution studied. Hence, a must be selected so that R_0 is near to R_c. In the case of the strong Boussinesq approximation, one finds $R_1=0$, thus $\epsilon \propto (R-R_c)^{1/2}$, which identifies the perturbation parameter ϵ in terms of the usual known quantities.

Solutions for a variety of planforms having been found, the question arises which is preferred in nature. To answer this, the stability of the solutions has been studied by the same techniques used to find them. The work is most complete for the case $\sigma \rightarrow \infty$ (Busse 1967a). In that limit the only known stable solutions are the two-dimensional rolls in a finite band of a, and even these become unstable at $R=22{,}600$. This instability represents excellent agreement with experiments which show that the motions do become three-dimensional at a value of R close to 22,600, but the theory does not predict the wavenumbers of the observed rolls. Also the experiments show the existence of steady three-dimensional motion for $R > 22{,}000$ and a challenge to the theory is to find a corresponding stable three-dimensional nonlinear solution. The observed cells have a rectangular appearance but theoretical rectangular planforms do not seem to be directly related to observed cells (Stuart 1964) and the motion probably consists of two interacting modes (Busse 1970, Busse & Whitehead 1971).

5. Dimensional Arguments

A powerful way to test ideas about convective transfer is to combine them with dimensional analysis to obtain the dependence of N on R and of $\overline{T}$ on z. There are two simple, but divergent, arguments about the form N should take, and they are worth outlining here.

The first argument stems from the observation that in highly developed convection at large R, $\overline{T}$ follows the adiabatic gradient over the bulk of the fluid. At the boundaries, according to 1.8, $\partial\overline{T}/\partial z = -N$, where N is a rather large number. For the case of fixed boundary temperatures, $\overline{T}$ must therefore change by an amount $(\Delta T - gd/C_p)$ in the thermal boundary layers near the walls. The total thickness of the layers is then $\sim d/N$. We may conclude that the ability to conduct heat into the main body of the fluid from the boundary layers is the limiting factor in fixing the heat transport and thus the structure of the boundary layer fixes N.

If this is true, what is the effect of changing d? One line of argument is that for highly active convection, a change in d should not affect the heat flux J, since it would not modify the boundary layers but merely increase the size of the intervening adiabatic region (Priestley 1954). Application of this argument to 1.7, 3.1, and 3.2 gives

$$N = A\sigma^{s}R^{1/3} \qquad 5.1$$

which is not incompatible with experiments at their present level of accuracy.

No information on Prandtl number dependence is provided by these arguments. But in stellar cases, where $\sigma \ll 1$, astrophysicists generally agree that the heat flux should not depend on viscosity, which implies that $s = r$ in 3.2 when $\sigma \ll 1$, i.e. $N = N(R\sigma)$. There is no experimental evidence directly supporting this conjecture except for the observed decrease of N with σ. Other theoretical arguments to be mentioned later also support this.

The other dimensional argument for the dependence of N or R comes from the formulation of methods used in stellar structure in the language of laboratory convection (Spiegel 1971a). Convection in stellar cores is treated as if the convective cores were completely adiabatic with no boundary or transition layers intervening between the convective regions and the radiative envelopes. The neglect of such layers implies that for the convective cores no uncertainty exists in the choice of the correct adiabat. It also assumes that whatever the luminosity of the model, the convective flux required will be carried without the limitation implied by the existence of a boundary layer with dominant conductive transport. Of course, stellar models are less constrained than laboratory models, since their dimensions are adjustable; nevertheless such physical requirements placed on the models are exacting and it is of interest to see what they imply. This is simple to do; if the convective zone is purely adiabatic the heat flux must be independent of thermal conductivity. This, taken with the previous conclusion that $N=N(R\sigma)$ for low σ, gives us

$$N = A(R\sigma)^{1/2} \tag{5.2}$$

if we demand that Equations 3.1 and 3.2 imply that J is independent of κ and ν.

The difference between 5.1 and 5.2 (even with $s=1/3$ in 5.1) is quite striking and its resolution is important to the theory. One possibility is that the stellar arguments should not be applied to the laboratory configuration. If that is true, then an important point of contact is lost between the two cases. However, other lines of argument which permit us to resolve the discrepancy between 5.1 and 5.2 will be given below. What is suggested by these arguments is that at sufficiently high R, turbulent breakdown of the thermal boundary layer occurs and causes a transition from 5.1 to 5.2. No such transition has been detected experimentally, but this is presumably explained by the limitation of the experiments to what in stellar terms are modest Rayleigh numbers ($R\sim 10^9$). The need to confirm (or deny) 5.2, which is intimately connected with basic ideas of stellar structure theory, poses a great challenge to the experimentalist.

Apart from these results, there have been attempts to discuss the structure of convective turbulence on the basis of similarity arguments (Zel'dovich 1932, Priestley 1959). Conclusions drawn about $\overline{T}(z)$ in this way do not seem to agree well with experiment (Townsend 1966).

6. Bounds on the Heat Transport

Integration of 1.8 over z leads to the expression

$$N = 1 + \langle w\theta \rangle \tag{6.1}$$

where the angular brackets denote a volume average. We may then apply the calculus of variations to this functional expression to place bounds on N which, if they are stringent enough, may be a useful guide in selecting among various theoretical results. The value of such bounds will evidently depend on the constraints added to the variational problem.

One set of constraints that has been used are the so-called power integrals (Malkus 1954b, Sorokin 1957, Chandrasekhar 1961). These are obtained by scalar multiplication of 1.4 by $\mathbf{u}$ and of 1.5 by θ followed by averaging over the fluid volume. If mean quantities are steady there result

$$R\langle w\theta \rangle = \langle |\nabla \mathbf{u}|^2 \rangle \tag{6.2}$$

and

$$\left\langle\left(\frac{\partial \overline{T}}{\partial z}+\frac{g}{C_p}\right)w\theta\right\rangle = -\langle|\nabla\theta|^2\rangle \qquad 6.3$$

The first of these expressions balances the rate of buoyant input of energy against the rate of viscous destruction; the second can be similarly thought of in terms of entropy generation. Neither of these conditions involves the Prandtl number.

Subject to 6.2 and 6.3 as constraints, the expression 6.1 for N can be maximized (Howard 1963) and it has been shown that

$$N \leq \frac{\sqrt{3}}{8} R^{1/2}$$

This bound shows an intriguing similarity in R-dependence to 5.2, but for $\sigma \ll 1$ it is much higher than 5.2. Attempts to further tighten this bound by the addition of 1.6 as a constraint (Busse 1969) have only modified the numerical coefficient in front of R^1
equations that then result are probably better representations of the actual flow.

Another bound has been found by replacing the constraint 6.2 by 1.4 in the limit $\sigma \rightarrow \infty$ and retaining 6.3 and 1.6 (Chan 1971). This gives $N \leq .325$ $R^{1/3}$ and suggests that for large enough σ, 5.2 cannot be correct. Thus, if as demanded by the astrophysical arguments, 5.2 holds for $\sigma \ll 1$, at fixed, large R, N should increase with increasing σ. At some unknown finite value of σ, $N(\sigma)$ should reach a maximum, $O(R^{1/2})$, and then decrease with increasing σ to an asymptotic value $\leq .325\ R^{1/3}$.

It seems technically feasible to derive similar bounds (though not with the same rigor) for astrophysical (non-Boussinesq) circumstances. However, the effort does not yet seem warranted. What is needed more is improvement of the bounds at low values of σ. Bounds for this case have been estimated by physical arguments (Spiegel 1971a), which are really dimensional, and which agree with 5.2. Analogous rigorous bounds have not been found, possibly for want of effort. However, there is another conceivable limitation on the closeness with which the bounds may approach the actual fluxes.

Suppose that the equations admit solutions to which correspond inordinately large heat fluxes, but that these solutions are unstable. Then it could happen that fluxes found in practice would be much lower than for these solutions but that rigorous mathematical bounds would not be. This difficulty might be circumvented by the use of ensemble means over many solutions or by an appropriate constraint; but the problem would be difficult. Such knotty problems notwithstanding, the establishment of further bounds would be helpful, especially on other quantities besides flux, such as the potential energy, or kinetic energy, of the fluid. Other less obvious quantities have been considered too (Busse 1967b).

7. Two-Dimensional Solutions

Naturally, one might try to solve the equations numerically, but even with the largest machines, complex flows in three dimensions have not been successfully treated except for R not much larger than R_c. In this region of low R ($<10\ R_c$) the three-dimensional time-dependent solutions evolve slowly to two-dimensional solutions, in confirmation of laboratory results (Chorin 1968). This does not mean that three-dimensional numerical calculations are not useful at low R; it turns out that if a two-dimensional flow wishes to adjust its horizontal scale, it does so through the action of three-dimensional perturbations (Lipps & Somerville 1971).

However, if the flow is two-dimensional, the Boussinesq equations can now be fairly routinely solved for R up to about 10^6 (Deardorff 1964, Fromm 1965, Plows 1968, Schneck & Veronis 1967, Somerville 1970, Veronis 1966), and no doubt these results can be pushed to higher values if adequate spatial resolution of the boundary layer can be achieved.

Such numerical calculations do not avoid the need to introduce a length scale in the horizontal direction since a finite horizontal dimension is required. Another limitation of two-dimensional solutions is that they do not seem to develop full turbulence. For reasons that are probably associated with this, the two-dimensional results show very little dependence on σ and hence are not directly useful to astrophysics except as a possible check on other theoretical approaches. Some σ dependence may be introduced by allowing the horizontal scale of motion to take its preferred value (Lipps & Somerville 1971), but the two-dimensional solutions have no way of determining this scale.

An especially interesting aspect of the two-dimensional problem is that asymptotic solutions for $R \rightarrow \infty$ have been obtained. These provide, apart from the static solution without convection, the only accurate solutions of the convection equations for large R. At large enough R, these are doubtless unstable, but they are of theoretical interest. In addition to having the kind of vertical structure in $\overline{T}$ already discussed, they have thin vertical "boundary" layers as well. For example in a horizontal line in the plane of the motion, $\overline{T}$ will be constant except for sharp bumps at the edges of the two-dimensional rolls. The vertical velocity is likewise concentrated in vertical layers.

For $\sigma \gg R^{3/5}$ the result $N \propto R^{1/5}$ is obtained (Roberts 1969) and $N \propto a^{2/5}$ for $a \gtrsim 1$ where a is the inverse horizontal scale that must be supplied to obtain a solution. For $\sigma \ll R^{3/5}$, the result $N \propto R^{1/4}$ is obtained (Wesseling 1969) but the σ dependence has not been given for this case. Also the corresponding flow exhibits separation in cell corners and this probably indicates instability. These results ($R^{1/5}$ and $R^{1/4}$) hold for rigid boundaries. For free boundaries one finds $N \propto R^{1/5}$ with very little σ dependence (Roberts 1969).

The analytic and numerical results (Veronis 1966) coincide in their lack of σ dependence. As we have mentioned, this limits their astrophysical value to tests of other techniques of solution or in exploring qualitative problems such as density variation. It is, however, becoming increasingly reasonable to think of solving three-dimensional problems even in the region of turbulent convection (Orszag 1969). Though this may be premature for the astrophysical case, it is a possibility that should be kept in mind in the coming decade.

8. Mixing-Length Theory

Most theories of convection consist of attempts to solve the basic equations in some approximation. Little effort has been made to bypass direct solution by constructing models for the flow which may lead to simpler equations (though some very interesting models for time-dependent convection exist (Chang 1957, Howard 1965, Keller 1966, Welander 1967, Elder 1968). A notable exception to this remark is the mixing-length theory which pictures turbulent transport processes in analogy with molecular processes (Taylor 1970, Prandtl 1952). A characteristic mixing length l, analogous to the mean free path of kinetic theory, and a characteristic turbulent velocity u', analogous to the mean molecular velocity, are introduced (e.g. Sutton 1955). Various formulations of the theory are possible, but the simplest is to introduce a turbulent diffusivity, or Austasch coefficient, lu', to be used to describe the turbulent transport processes. This part of the theory is reasonably clear; the choice of l and u' is more difficult. The random velocity u' is often taken to be of the order of some large-scale velocity in the fluid; in convection this is the vertical velocity w. The mixing length is taken to be a characteristic scale which is sometimes a constant such as the size of the system. More usually it is assumed to be a local scale such as the distance to a boundary or the scale of variation of a dynamically important quantity, such as velocity, shear, pressure, or density. These choices imply a slight inconsistency since the transport is described as a diffusion, using the approximation that l is less than any characteristic scale of motion (Spiegel 1963).

The equations describing the mixing-length model are just 1.4, 1.5, and 1.8 without the $\mathbf{u} \cdot \nabla \mathbf{u}$ and $\mathbf{u} \cdot \nabla \theta$ but with terms added to account for the turbulent diffusion. In the simplest case, the same diffusion coefficient wl is used for all quantities such as temperature and velocity, though it may be more correct to consider different values of l for different quantities. This is done in other problems such as vorticity diffusion (Goldstein 1938).

Even the relatively simple mixing-length equations are difficult to solve and, in view of the drastic approximations involved already, perhaps the effort needed to find exact solutions is not called for. Hence further mathematical approximations are usually introduced. In oceanography, for example, one often sets wl constant. In the astrophysical examples, one typically replaces most spatial derivatives of fluctuating quantities by l^{-1}; for example $\nabla^2 \sim l^{-2}$ (Kraichnan 1962). An equivalent way to obtain such results is to be more explicit about the model itself and to picture the turbulent transport as being effected by parcels of fluid of size l before disruption (Vitense 1953, Spiegel 1971b).

If we proceed by the expedient of writing l^{-1} for derivatives, lump together terms that seem to be of comparable magnitude, and drop the pressure and time derivatives, we can write by inspection from 1.4 and 1.5,

$$g\alpha l\theta = \nu(\mathrm{Re} + 1)w \qquad 8.1$$

and

$$w\left(\frac{\partial \overline{T}}{\partial z} + \frac{g}{C_p}\right) l^2 = \kappa(\mathrm{Pe} + 1)\theta \qquad 8.2$$

where

$$\mathrm{Pe} = \frac{wl}{\kappa}, \qquad \mathrm{Re} = \frac{wl}{\nu} \qquad 8.3$$

These equations have been written here in dimensional form since that is the usual practice in the literature, and are to be solved in connection with 1.8.

The dimensionless ratios Pe and Re are known as the Peclet and Reynolds numbers and measure the ratio of turbulent diffusivity to the two molecular diffusivities of the convection problem. Convection differs from other turbulence problems in that no velocities are externally prescribed, hence Pe and Re are derived rather than imposed quantities. It is possible to make estimates for them if we take for w a characteristic free-fall time through the fluid, namely

$$w \sim [g\alpha(\Delta T - gd/C_p)d]^{1/2} \qquad 8.4$$

Then

$$\mathrm{Pe} \sim (\sigma R)^{1/2}, \qquad \mathrm{Re} \sim \left(\frac{R}{\sigma}\right)^{1/2} \qquad 8.5$$

(These estimates provide a qualitative guide, and more precise values differ with differing states of motion.) This shows that under astrophysical conditions we expect convection to be turbulent since $\mathrm{Re} \gg 10^3$, but that in stellar envelopes where κ may be large, turbulent transfer does not necessarily dominate radiation transfer everywhere since $(\sigma R)^{1/2}$ is not always large.

A few further steps are needed to put these equations in the usual form used by astrophysicists. We introduce $\alpha^{-1} = \overline{T}$, which holds for a gas. Let

$$\theta = l\frac{\partial \theta}{\partial z} = l\frac{\partial}{\partial z}(T - \overline{T}) \qquad 8.6$$

and take $\mathrm{Re} \gg 1$ everywhere. Then the equations are readily rearranged into the form used for stellar models with the notation $\Gamma = \mathrm{Pe}$ (Vitense 1953). There are some differences in numerical coefficients, such as those which arise when a more flexible approximation to the radiative diffusion term is used. These matters aside, it is clear that the equations presently used for stellar convection are essentially those used for laboratory convection in the mixing-length approximation.

Leaving for Part III the question of stellar application and related modifications of the theory, we may summarize the results indicated by the mixing-length theory for laboratory convection (Kraichnan 1962). If careful account is taken of the boundary-layer structure, the Nusselt number for σ large enough ($\sigma > 0.1$) and R quite large becomes $N \simeq (R/1500)^{1/3}$, where the constants in this and the succeeding results are estimated from a variety of empirical data and theoretical calculations. For low σ (<0.1), $N \simeq (\sigma R/70)^{1/3}$, where σR must be large (>300).

An important feature of the standard mixing-length theory as described here is that it includes the effect of small-scale motions only through their damping of the large-scale motions. In turn, these large-scale motions are driven only by differential buoyancy forces, and the mixing-length theory chooses at each position a preferred scale of motion l. Near the boundaries, l becomes small and at some distance from the boundary, a local Rayleigh number computed with l instead of d is $\sim 10^3$. For distances to the wall less than this, no motions can be strongly excited by the buoyancy, and conduction becomes the dominant mode of transport. This distance then is the thickness of the thermal boundary layer which we saw in Section 3 is $\sim d/N$. Hence N is found by saying that the Rayleigh number computed for a scale d/N is R/N^3 and this should be $\sim R_c$. From this we find $N \sim (R/R_c)^{1/3}$. A modification must be added in mixing-length theory since the diffusivities entering into R may be turbulent diffusivities. In this model the thermal boundary layer is the region in which molecular (or radiative) conduction dominates over turbulent conduction. However, when the Prandtl number is small the turbulent viscosity may be larger than the molecular viscosity even in the thermal boundary. Hence, in general, the Rayleigh number of the thermal boundary layer, R/N^3, should be corrected for turbulent viscosity. That is, the Rayleigh number of the thermal boundary layer needs a corrective factor $\nu(\nu + wl)^{-1}$ (Spiegel 1967) which is to be evaluated at the edge of the boundary layer; at high Prandtl number this factor becomes unity. If l is proportional to the distance from the boundary layer, we must take $l = d/N$ and in analogy to 8.4 we take $w \sim [g\alpha\Delta T\, d/N]^{1/2}$ as the value for the edge of the thermal layer. (In the laboratory case we may neglect the correction for the adiabatic term gd/NC_p.) We find

$$N \sim \left[\frac{\sigma R/R_c}{\sigma + (\sigma R/N^3)^{1/2}}\right]^{1/3} \qquad 8.7$$

More detailed arguments, allowing for geometrical factors, suggest that in the denominator, R should be replaced by R/R_c. We then find

$$N \sim \left[\frac{\sqrt{1+4\sigma}-1}{2\sigma}\right]^{2/3}\left(\frac{\sigma R}{R_c}\right)^{1/3} \qquad 8.8$$

as the mixing-length prediction for N. This has the limits mentioned; in particular for low σ, $N \sim (\sigma R)^{1/3}$, which corresponds to the astrophysical notion that heat transport should be independent of viscosity for low σ. At very large σ, 8.8 indicates that the slight dependence of N on σ is not well represented by a power law, but rather by a factor like $(1-\text{const}\ \sigma^{-1/2})$.

At large σ, these mixing-length results agree reasonably well with experiment, in the sense that the power r in 3.2 seems to be tending toward 1/3 experimentally. The lack of σ dependence for large σ in the mixing-length results is also not a bad representation of the data. For low σ, data are available only for mercury and these are not adequate for a real test, but the data for σR large do seem consistent with the mixing-length predictions. Attempts to compare mixing-length predictions for $\overline{T}$ and $\overline{\theta^2}$ with experimental results are qualitatively acceptable for large σ but are unsatisfactory for mercury ($\sigma = 0.025$) (Rossby 1969). The values of $(\sigma R)^{1/2}$ studied experimentally for mercury are, however, not very large and the disagreement may not be wholly damning to the theory. This point needs further experimental scrutiny.

The standard mixing-length theory, as we have seen, considers the heat transport by motions driven by differential buoyancy forces. The motions which provide turbulent diffusivity are presumably dynamically excited by the $\mathbf{u} \cdot \nabla \mathbf{u}$ terms in the equations of motion. The effect of these motions is normally included only as a drain on the large-scale motions, but their contribution to the convective transfer should also be included (Kraichnan 1962). In the interior of the fluid such corrections are unimportant since the gradient is already nearly adiabatic, but they can be of great importance in the boundary layer.

The reason for the frequent omission of such corrections is that standard mixing-length theory asserts that at any location a specific scale of motion l (normally = distance from the boundary) is dominant. Thus, the usual mixing-length theory would not have predicted that large eddies from the interior of the fluid strike the boundaries, and set up appreciable horizontal motions there. There is, however, evidence that this occurs in laboratory convection (Malkus 1954a) and it seems to be manifested on the Sun as supergranulation (Noyes 1967; Simon & Weiss 1968). The neglect of the transport at the boundary by these large-scale motions is justified normally since they do not carry large temperature fluctuations, having traveled mostly through nearly adiabatic regions. What cannot always be neglected is the dynamical effect of these motions.

When large eddies hit the boundaries they set up shear layers which, when they are intense enough, can break down into small-scale turbulent motions. This turbulence may be less intense than the turbulence usually included in mixing-length theory, and it will not in general produce as large an eddy conductivity. But if this turbulent conductivity exceeds the molecular conductivity a marked change in boundary-layer thickness occurs. It must be understood, however, that there is still a thermal boundary layer, whose edge now is at the place where the large eddy conductivity of buoyancy-driven turbulence is no longer important and the weaker conductivity of shear turbulence takes over. This boundary layer is a turbulent boundary layer which permits a greater heat transport than the laminar boundary layer. The details of this layer depend on the boundary conditions and the effect of the boundary-layer turbulence has been estimated only for the laboratory case (Kraichnan 1962).

The procedure is to obtain the velocity shear at the wall by assuming a simple model for the eddies arising from the interior. The intensity of the resulting turbulence driven near the wall may be computed in analogy with the results available from work on shear turbulence. Such mechanically driven motions in the presence of a temperature gradient would transport heat (Prandtl 1952) and, as in ordinary mixing-length theory, the additional transport may be estimated. The details of the calculations are too lengthy for inclusion here. They give, for sufficiently large R,

$$N \sim \left(\frac{\sigma R}{(9 \ln R)^3}\right)^{1/2} \qquad 8.9$$

when σ is small. (A more complicated result for large σ is obtained, but the main factor in the expression for N in that case is $R^{1/2}$.) Apart from the logarithmic term, 8.9 is in substantial agreement with 5.2.

A sensitive question is: when does 8.8 give way to 8.9 at small fixed σ, as R increases? Unfortunately, the estimates for this rely heavily on high powers of badly known constants. It appears that for the case $\sigma \sim 10^{-9}$, the transition from 8.8 to 8.9 begins at $R \sim 10^{24}$. Thus, these corrections for boundary-layer turbulence do not seem required under solar conditions. Whether they may be required under any conditions in stars is an open question since these turbulent corrections cannot be confirmed even for the laboratory case with existing data. Nevertheless, such corrections may well be needed for stellar convection (Spiegel 1971a) and the problem will be further discussed in Part III.

Perhaps the most satisfying aspect of the calculations with boundary-layer turbulence is that they permit a rationalization of the difference between 5.1 and 5.2. The suggestion is that for large R, 5.1 is a reasonable representation until the large-scale motions become turbulent in the thermal boundary layers associated with 5.1. The point at which this happens depends on R in a moderately complicated way. Once the boundary-layer turbulence starts, the heat transfer should then depend on d since it is an important factor in the large-scale interior velocities, as 8.4 indicates. Thus at some enormous value of R the heat transport will depend on d and indeed it is roughly found from $J \sim w\theta$ where w is given by 7.4 and $\theta \sim \Delta T$.

In this new regime, the flux no longer depends on κ, except perhaps logarithmically. This comes about because what was previously a laminar thermal layer becomes a turbulent layer and the role of molecular conductivity is taken up by turbulent conductivity. There may be in the new boundary layer a small laminar sublayer which accounts for the logarithmic terms, but without solid boundaries even the sublayer might disappear (Clauser 1961). Of course there are more details involved, particularly for $\sigma > 1$, but the main qualitative features seem to be consistent with the dimensional arguments.

9. Truncated Expansions

A surprisingly effective approach to solving the equations of convection is the use of truncated expansions in terms of an appropriate set of basis functions or modes. Such a procedure is often called a *Galerkin method* (Reiss 1965) though the terminology used varies depending on whether the basis functions depend on one or more of the independent variables. There exist other related approximation methods, especially those relying on variational approaches (Reiss 1965, Finlayson & Scriven 1966), which, in the convection problem, have not proved as simple to use (Roberts 1966). In the Galerkin and related methods one expands in functions of one or more of the independent variables to obtain an infinite set of coupled equations for the amplitudes in the expansion. Expansion in functions of space and time, of only space coordinates, and of only the vertical coordinates have been considered, with varying degrees of success, depending on the choice and number of basis functions, and the techniques used in solving the reduced equations. The greatest effort, however, has gone into the use of basis functions depending on only the horizontal coordinates. Here too a wide choice of basis is possible, but a promising set is comprised of the planform functions of linear theory. Their relevance is reinforced by the theoretical suggestion that cell shape is preserved even for $R > R_c$ (Stuart 1960). The planform functions satisfy 2.3 and contain as a special case the trigonometric functions. They have the useful property that for two different wavenumbers a_i and a_j the corresponding solutions of 2.3, f_i and f_j, satisfy $\overline{f_i f_j} = 0$. To each a_i there corresponds a subspace of f_i, and strictly we should add a second index to indicate the various members of this subspace. But to keep the formulae simple, we shall merely let f_i represent the most general linear combination of basis functions associated with a_i where $\overline{f_i^2} = 1$. Then we can write the expansions for w and θ

$$w(\mathbf{x}, t) = \sum_i f_i(x, y) W_i(z, t); \qquad \theta(\mathbf{x}, t) = \sum_i f_i(x, y)\Theta_i(z, t) \quad 9.1$$

with similar expansions for u, v. Here, the index i is treated as discrete to avoid the difficulty of infinite norm associated with a continuous spectrum.

The expansions may be introduced into 1.4, 1.5, and 1.8 and the expanded equations projected onto the appropriate f_i in the usual way. The resulting equations for the amplitudes in the expansions can be distilled into equations for W_i, Θ_i, and $\overline{T}$. For simplicity we shall here suppress terms describing vertical components of vorticity; these do not alter the general form of the equations, which are (Gough, Spiegel & Toomre 1971),

$$\left[\frac{1}{\sigma}\partial_t - (\partial_z^2 - a_i^2)\right](\partial_z^2 - a_i^2)W_i$$

$$= -Ra_i^2\Theta_i - \frac{1}{\sigma}\sum_{j,k} C^{ijk}[W_k L_{kij}\partial_z W_j + (\partial_z W_k)M_{kij}W_j] \quad 9.2$$

$$[\partial_t - (\partial_z^2 - a_i^2)]\Theta_i$$

$$= -\left(\partial_z\overline{T} + \frac{g}{C_p}\right)W_i - \sum_{j,k} C^{ijk}(A_{ijk}\Theta_j\partial_z W_k + (2a_i^2 a_k^2)W_k\partial_z\Theta_j) \quad 9.3$$

and

$$\partial_t\overline{T} + \sum_j W_j\Theta_j = \partial_z^2\overline{T} \quad 9.4$$

where

$$\partial_z = \partial/\partial z, \qquad \partial_t = \partial/\partial t$$

$$2C^{ijk} = \overline{f^i f^j f^k} \quad 9.5$$

$$L_{ijk} = A_{ijk}(\partial_z^2 - a_k^2), \qquad M_{ijk} = L_{ijk} + \left(\frac{a_j}{a_k}\right)^2 L_{jki} \quad 9.6$$

and

$$A_{ijk} = \frac{1}{a_k^2}(a_j^2 + a_k^2 - a_i^2) \quad 9.7$$

The basic feature of these equations is that the amplitudes, or modes, are coupled in two ways. There are the direct or dynamical couplings whose strengths are mediated by the coupling constants C^{ijk}. These interactions give rise to the disorder in the flow that is the hallmark of turbulence. The other form of modal interaction, the so-called *mean-field interaction*, comes from the term $(\partial_z\overline{T})\ W_i$ in 9.3. In turn, $\partial_z\overline{T}$ is given by 9.4, which shows how $\overline{T}$ is affected by the motion. Thus all the modes may interact through the mean temperature.

Consider now the most drastic truncation, in which only one mode is retained. The resulting equations can be integrated numerically, and for a wide range of parameters and initial conditions the solution tends to a stationary state. However, for a given choice of the parameters, there is not a unique stationary solution, there being nonlinear analogues of the fundamental and overtones (in z) of linear theory. For the one-mode case, we shall confine attention to the fundamental mode.

For one steady mode the nature of the solution depends on whether the self-coupling constant, $C^{111}=C$, vanishes or not. If f_1 contains at least three Fourier components whose wavevectors form a triangle, we will have $C\neq 0$. Evidently, for rolls or rectangles we will have $C=0$, and this case has been studied extensively (Herring 1963, 1964, 1966). For $R\rightarrow\infty$ one finds (Roberts 1966, Stewartson 1966)

$$N_1 \sim 0.28\left(1-\frac{a_1^4}{R}\right)^{6/5}\left[Ra_1^2 \ln (Ra_1^2 - a_1^6)\right]^{1/5} \qquad 9.8$$

for rigid boundaries and for $a_1<R^{1/4}$. (For $a>R^{1/4}$ no convective solutions exist.) As a function of a_1 this expression has a maximum at $a_{1\,\max}\sim(R/13)^{1/4}$, and for most purposes we need consider only $a_1\leq a_{1\,\max}$. Hence to good approximation

$$N_1 \simeq A_1[Ra_1^2 \ln (Ra_1^2)]^{1/5} \qquad 9.9$$

It is interesting that as in the two-dimensional problem, the boundary conditions are very important; for free boundaries $N\propto R^{1/3}$ (Howard 1965, Herring 1966).

For cases when $C\neq 0$ but $a_1<a_{1\,\max}$, expression 9.9 continues to hold for $R\rightarrow\infty$ except that A_1 is no longer ~ 0.28 but is a function of C and σ (Gough, Spiegel & Toomre 1971). The functional form of A_1 is not known analytically, but for $C/\sigma<1$ we have $A_1^5\sim[3/5(2/\pi)^6]$ while for $C/\sigma>1$, $A_1^5=3/5(2/\pi)^4\sigma/C$. Thus, the introduction of the self-interaction terms causes N_1 to depend on σR for small σ.

Given these results we are left with the problem of choosing the a's and C's for one or more modes. Ideally, we would like to use the experiments as a guide, but they do not directly provide scale information at large R. Nevertheless it is possible to let a_1 and C be functions of R and σ and match the observed N over a fairly wide domain of R and σ even with one mode. However, at $a_{1\,\max}=0\ (R^{1/4})$, $N_{1\,\max}\propto R^{.3}(\ln R)^{.2}$, while for large enough R, the actual N goes up like $R^{1/3}$ (or faster). Hence one mode can never describe the full behavior of N. Further modes will be needed at very high R and no experimental indication of which set of modes to choose is evident.

In the absence of stringent experimental guides, the choice of modes must be dictated by additional assumptions. One that has been used is to maximize N with respect to a and C. For moderate values of R this gives much too high a heat transport. Alternately, one might take over an idea that has been preferred by many astrophysicists and choose that mode which is most unstable according to linear theory. For large R, η, the growth rate of linear theory is maximum when $a_1\propto R^{1/8}$ and this gives $N_1\propto R^{1/4}(\ln R)^{1/5}$, which is like the law suggested by some experiments for laminar convection at moderate values of $R(10^5-10^6)$.

These two methods of selecting modes suggest that a_1 should increase with R while in fact the experiments (at least for $R<10^5$) indicate a decrease (Koschmieder 1969). Hence, perhaps the most reasonable procedure is to take $a_1=0(1)$ and to represent the more rapid increase of N with R at large R by adding further modes. Of course, this means that the choice of further modes is left open as well, and one must seek an extension of this procedure.

The first possibility to consider when adding higher modes is to add to the basic mode successively more of its horizontal harmonics to obtain an accurate description of cellular motion. If, for example, we start with a roll, $f_1=\sqrt{2}\sin(a_1x)$, and its harmonics, we will have a Fourier representation of nonlinear, two-dimensional cellular convection with horizontal periodicity a_1^{-1}. We have seen that solutions of the two-dimensional problem for $\sigma>R^{3/5}$ yield $N\propto R^{1/5}$. Hence the addition of higher harmonics does not produce a drastic change in the heat transport calculated from one mode (at least for rolls at high σ) and this implies that the vertical structure of mean quantities such as $\overline{T}$ as given by one mode is a reasonably close approximation to a solution of the full equations. (The agreement is even better for free boundaries in which case one mode gives $R^{1/3}$.) On the other hand, the horizontal structure implied by the one-mode approximation is generally not at all like that of the full two-dimensional solution, which suggests the existence of rising and descending plumes. Thus, the modal expansion may well be useful for computing mean structures in a convection layer in spite of its gross misrepresentation of the horizontal variations.

If, as is suggested by this comparison, the lower harmonics are not the most important additional modes to introduce, which are? Presumably, the rule used to choose the first mode should be used to choose successive modes. Thus the second mode might be that which is most unstable according to linear theory, based on the conditions existing with one mode. Or the second wavenumber a_2 might be chosen to be of order unity in terms of the vertical-length scale introduced by the first mode; this would imply $a_2=O(N_1)$ since the boundary layer has thickness d/N_1. We should also note that once two modes are retained, another option is open: the second mode may be used as a perturbation to study the stability of the first mode. In principle this may be a way to help select the preferred first mode, but in practice if the nonlinear stability problem with two modes is studied, no clear choice is indicated. We must therefore adapt one of the ad hoc selection procedures and this is a weakness of the Galerkin procedure in convection. That the difficulty becomes less serious as more modes are introduced may be seen by estimating N_n, the Nusselt number computed from n modes.

Consider first the case of only Fourier modes; these have $C^{iii}=0$. Then if we add more and more modes, so long as they are not harmonically related, we would obtain an approximation to Equations 1.4–1.8 with the terms $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ and $\boldsymbol{u}\cdot\nabla\theta-\overline{\boldsymbol{u}\cdot\nabla\theta}$ neglected. The neglect of these terms is often called the *mean-field* or *weak-coupling approximation* (Spiegel 1967). It can be expected to hold only for very large σ as indicated by the agreement of the one-mode roll solution and the two-dimensional theory. In that limit, the $\overline{\boldsymbol{u} \cdot \nabla \boldsymbol{u}}$ term is negligible, but there seems no obvious reason to omit $\boldsymbol{u} \cdot \nabla\theta - \overline{\boldsymbol{u} \cdot \nabla\theta}$. Of course, this term vanishes for one mode with $C=0$, and since the observed motion is two-dimensional for large σ and $R \lesssim 10\ R_c$, the approximation should work well in that domain of parameter space. At higher R it should continue to hold approximately since the term $w\partial_z \overline{T}$ acts qualitatively like $\boldsymbol{u} \cdot \nabla\theta$. Part of the reason for this is that for $\overline{T}$ independent of time the statistically steady solutions of the mean-field equations must be stationary (Spiegel 1962a).

To estimate the effect of higher modes we assume that a rule for choosing the first mode has been adopted. In general, for large R, such a rule may be expressed as $a_1 = k_1 R^{\mu}$, where $\mu \leq 1/4$ and k_1 may depend on σ. (We might also include a factor involving ln R.) Then, for this mode, we have from 9.9,

$$N_1 = AR^{\alpha}(\ln R)^{1/5} \qquad 9.10$$

where $A \simeq A_1(5\alpha k_1^2)^{1/5}$ and $\alpha = 1/5\ (2\mu+1)$. For the second wavenumber let us adopt the same prescription, based on the length unit d/N_1; then $a_2 = k_2 N_1 R_{eff}^{\mu}$, where the effective Rayleigh number R_{eff}, seen by the second mode is, crudely, $\sim R/N_1^3$. Here an arbitrary factor of order unity should be included to allow for the deviation from linearity of $\overline{T}$ in the boundary layer and for the lack of a rigid boundary at the edge of the boundary layer. The ratio k_2/k_1 differs from unity for similar reasons. Let us ignore such corrections and take $R_{eff} = R/N_1^3$ and $k_2 = k_1$. Then the effect of the second mode is to increase N by an approximate factor $A\ R_{eff}^{\alpha}(\ln R_{eff})^{1/5}$ and for the two modes, the product of this factor with N_1 gives

$$N_2 = A^{2-3\alpha}(1-3\alpha)^{1/5} R^{(2-3\alpha)}(\ln R)^{1/5(2-3\alpha)} \qquad 9.11$$

We may readily extend this procedure to n modes with the result

$$N_n = (1-3\alpha)^{n/(15\alpha)} \left\{ A^{1/\alpha}(1-3\alpha)^{-1/(15\alpha^2)}(\ln R)^{1/(5\alpha)} R \right\}^{[1-(1-3\alpha)^n]/3} \qquad 9.12$$

We see that as n is increased, the chief feature of the limiting N is a variation like $R^{1/3}$, except for logarithmic factors, and that this behavior does not depend on α. However, for fixed R, N_n tends to zero as n tends to infinity, which shows that some cutoff must occur if the result is to be meaningful. That such a cutoff exists is shown by numerical solutions which indicate that if a_n is too large, the nth mode will not develop a detectable amplitude. The numerical results are consistent with the criterion that $R_{eff} = R/N_n^3 \sim R_c$ for the cutoff n. This criterion not only permits us to estimate $n \sim -\ln \ln R/\ln(1-3\alpha)$, but it also shows that $N \sim (R/R_c)^{1/3}$, irrespective of the details of the theory (Malkus 1954b). Another approach, in the case where one wishes to maximize N, is to choose n to maximize N_n for fixed R (Chan 1971). Curiously, this procedure gives the same cutoff for large R, so that for the mean-field approximation $N_{max} \propto R^{1/3}$. For the case of maximum N, accurate asymptotic solutions of the mean-field equations have recently been constructed (Chan 1970).

The recovery of the $R^{1/3}$ law is an encouraging result, and the lack of σ dependence is not surprising since the mean-field equations should hold only at large σ. We must now consider more general modes with $C \neq 0$. These do introduce a σ dependence. If we proceed again with the introduction of a sequence of modes of higher and higher wavenumber, we will again come to the result 9.12 as long as the modes are not harmonically related so that their mutual coupling constants vanish. This time A will depend on σ in a way that depends on our choice of wavenumbers. If at small σ we expect to have N depend on $R\sigma$ we must choose k_1 so that at small σ it varies like σ^{μ}. In that case, we shall find that $N \propto (R\sigma)^{1/3}$ for large $R\sigma$ and for any $\mu < 1/4$. Thus the truncation procedure gives us the same kind of result for heat transport as standard mixing-length theory. Its advantage over mixing-length theory is that it can accommodate variable density, time dependence, and many of the other features of the problem that must be dealt with in stellar convection. Many of these complications have already been studied and will be discussed in Part II.

Now we must ask: what happens when we deal with dynamically coupled modes in the truncation theory? In particular, do the dynamical couplings excite modes that would not have been excited by mean-field terms and do these excitations lead for example to an $(R\sigma)^{1/2}$ law at large $R\sigma$? The question has not yet been answered. Numerical solutions for R up to 10^9 with three dynamically coupled modes have been found. These do seem to indicate the possibility of dynamical excitation. They also introduce complicated time dependence into the solution (Toomre 1969). However, the value of $R(=10^9)$ achieved to date is not high enough to show a tendency to deviate from an $R^{1/3}$ law; indeed at that value of R the law is still being approached as R increases.

One difficulty with going to higher R is that if we seek steady solutions, there is at any R and σ for a given set of wavenumbers and coupling constants a great wealth of solutions and it is not clear which solution branch to follow. It is just too demanding of computing time to follow them all. On the other hand, even if the difficulty of choice can be alleviated by computing time-dependent solutions, the computing bill also mounts up quickly because of the long transients. The time-dependent problem is reminiscent of computations in stellar pulsation in having one space and one time dimension; however, for three modes we must deal with a system of 20th order in the spatial derivatives. The problem can be done with existing machines, but it is difficult. It may well be possible to extract the main results for dynamically coupled modes analytically as has been done for the modes with only self-coupling and mean-field interactions. This problem has not been attempted and it is clearly of great analytical complexity.

An alternate procedure is to keep just a few (one to three) modes and add terms based on turbulence theory (crude or refined). This would permit a reasonably accurate handling of many of the features of stellar convection.

10. Turbulence Theories

The development of the statistical theory of turbulence has been proceeding quite rapidly, the recent developments being largely dominated by improved approximations for highly turbulent flows (Kraichnan 1970, Orszag 1970, Saffman 1968). The approximation techniques are usually developed in terms of turbulent spectra and produce equations, which, though much easier to solve than the original equations of motion, pose great calculational difficulties. In particular, one such approximation scheme, the direct interaction, has been applied to the Boussinesq equations for convection (Kraichnan 1964). The numerical solution of these equations has recently been accomplished for the case of free boundaries with $R \leq 10^4$ and $\sigma = \infty$ (Herring 1969). Remarkably enough the solutions are almost identical to those obtained for the mean-field equations with one mode. The extension of these solutions to higher R and lower σ is difficult, but feasible, and it will be of great interest to see how far such extensions can be carried. It should be stressed that this approach leaves no arbitrary parameters.

The use of approximations from turbulence theory has also been applied to the case of very low σ with rather bewildering results: the solutions grow in amplitude without limit (Herring 1970). This may be a peculiarity of the free-boundary conditions used in connection with a particular limiting form of the Boussinesq equations for low σ (Spiegel 1962b), but the matter has yet to be resolved.

There has also been a renaissance of phenomenological theories, many of which are more sophisticated than the standard mixing-length theory (Crow 1968, Lumley 1970, Nee & Kovasznay 1969, Parker 1969, Saffman 1970). The term phenomenological is sometimes used pejoratively in turbulence theory; here it is not. It simply implies that the approach used is not intended to be completely deductive and is based to varying degrees on some physical picture of the inner workings of turbulence. Nor does it follow that the equations in a phenomenological approach are easy to solve, though usually they are easier to solve than the full equations. The disadvantage of these theories for astrophysical purposes is that, like mixing-length theory, they normally contain disposable parameters. Values for these parameters determined in the laboratory may not be applicable to stars. But this is a question to be faced when these new approaches are applied to convection; for the present they simply represent a trend of which the astrophysicist need only be aware.

11. Conclusion

Not everyone who works on convection would agree with the assessments made of the various approaches discussed here. But the discussion as given does seem to bring out certain conclusions and these are:

1. The mixing-length theory, for all the criticisms leveled at it, has been qualitatively successful in predicting convective heat transfer. The form used for stellar convection however seems to be incomplete for very strong convection, and additional ingredients will be suggested in Part III. In any case it is difficult to use the theory for many special effects of astrophysical interest.

2. The procedure of maximizing the heat transfer is quite promising and gives Euler equations that seem to represent the flow qualitatively. Hitherto, these Euler equations have not contained dynamical couplings and this is the key to their tractability. If constraints could be added to the maximization problem which bring out the Prandtl-number dependence, the method will have great promise for stellar convection. Whether this can be done without enormous complications in the Euler equations remains to be seen.

3. The truncated modal expansions provide a reasonably accurate and flexible approach to the problem. They can readily accommodate large density variation, time dependence, and other difficulties of the stellar case. As yet, for more than one mode, they must be solved numerically, if the dynamical couplings are to be treated with acceptable accuracy. Whether these couplings will adequately describe boundary-layer turbulence remains moot. To settle this it may yet be necessary to further approximate the dynamical couplings.

4. The statistical theories of turbulence are beginning to provide dividends and will certainly give answers in the next few years. These results are awaited eagerly although they will oblige interested parties to master this very difficult discipline.

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